Control Systems

Lecture: 4

Topics Covered

- Introduction to Nyquist plot
- Stability Analysis

Nyquist Criteria Useful

- Determine Stability
- Determine Gain & Phase Margins
- ‘Medium’ effort. Finds **number** of RHP poles of T(s), the closed-loop transfer function.
- Does not find pole values explicitly. (Similar to with Routh-Hurwitz).

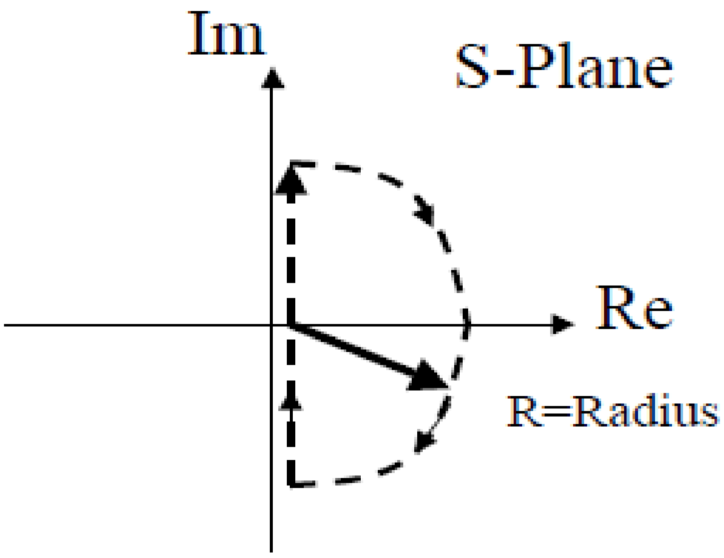
Define F(s) = Denominator of T(s)

- $T(s) = KG(s) / [1 + KGH(s)]$
- $F(s) = 1 + KGH(s)$
- T(s) is stable iff zeros of F(s) are in LHP.
- Note:

  Zeros of F(s) are ______ of T(s), which are <u>hard/easy</u> to find.

  Poles of F(s) are ______ of KGH(s), which are <u>hard/easy</u> to find.

Consider Contour in S-Plane

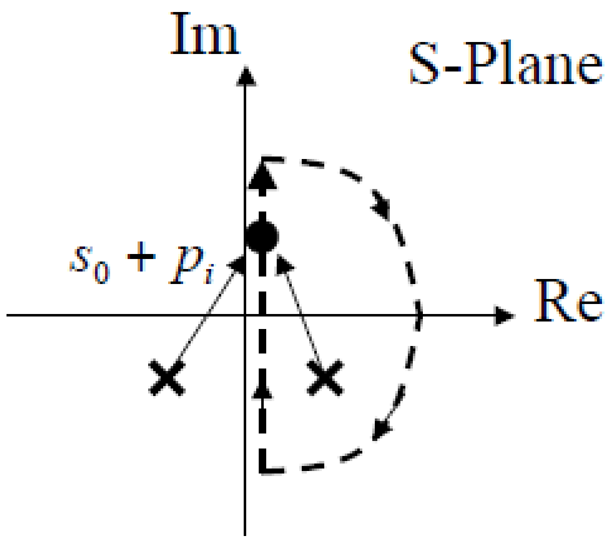


- Contour travels up jw axis.
- Contour encircles all RHP poles/zeros.
- |R| -> infinity.
- Assume for now: *No poles on jw axis*(*)

* Integration along contour must avoid infinite values (rude)

Consider Net Phase Change To Factors of F(s)

$$F(s) = \frac{(s + z_1)\cdots(s + z_M)}{(s + p_1)\cdots(s + p_N)}$$



- Point s=s0 moves around contour CW direction.
- Vector differences (s0+pj) experience change in phase angle.
- *What is the accumulated phase contribution to <F(s) from <(s0+pj), as s0 traverses contour?*

???	RHP	LHP
Zero		
Pole		

Imagine (s0+pj) to be a handle of a crank winding a spring…
Integration of phase <(s0+pj) along contour analogous to winding spring

Number of RHP Poles & Zeros Are Revealed by Net Phase Change

- Define

Z = # RHP Zeros of F(s) = # RHP Poles of T(s) [*Something we want to know*]

P = # RHP Poles of F(s) = # RHP Poles of KGH(s) [*Something easy to find*]

N = <u>(Net phase change in F(s) as s traverses contour CW)</u>

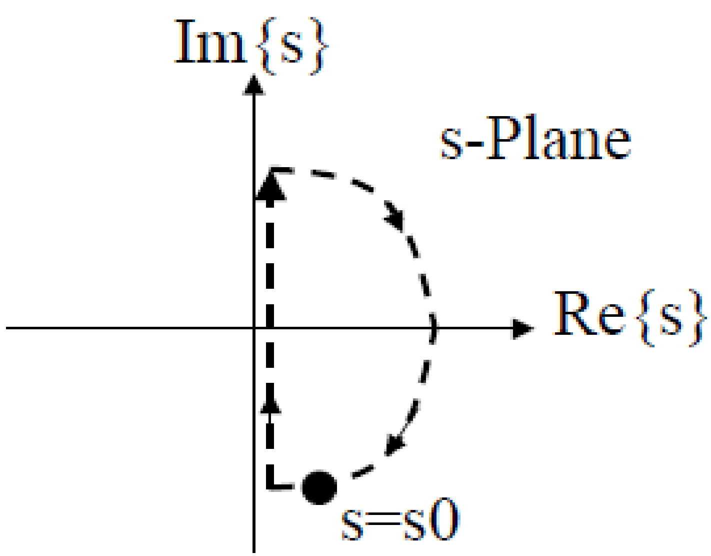
-360 Degrees

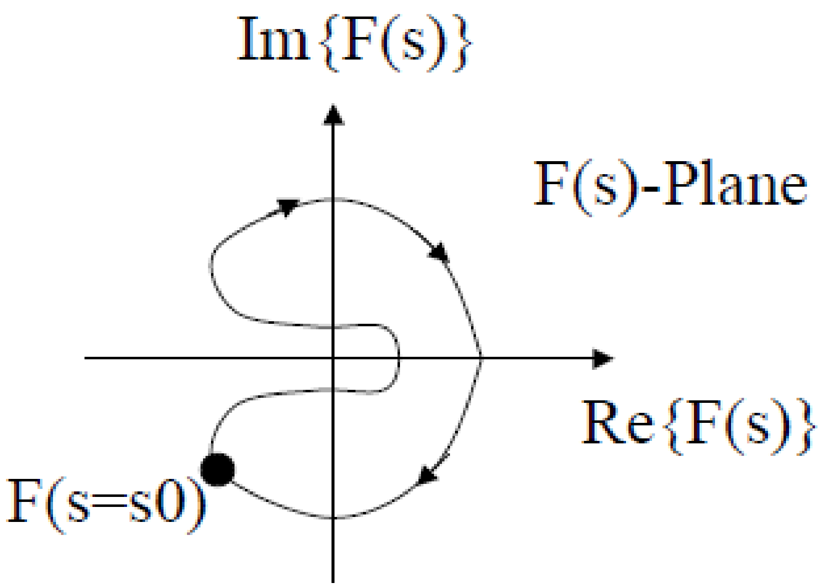
- Example
  - If P=0, Z=1 Then N = ___
  - If P=1, Z=0 Then N = ___
  - If P=1, Z=1 Then N = ___
- Relation Between N,Z,P? __Z = ________

New Representation: F(s)-Plane

- Plot F(s) as s varies along contour.
- Phase of F(s=s0) directly observable from plot.

  Consider polar form of F(s), a 'polar plot'.
- Accumulated phase change of F(s) directly observable from plot. What is the criteria for N=1? ________________
  - N = (Net phase change in F(s) as s traverses contour CW) / -360 Degrees

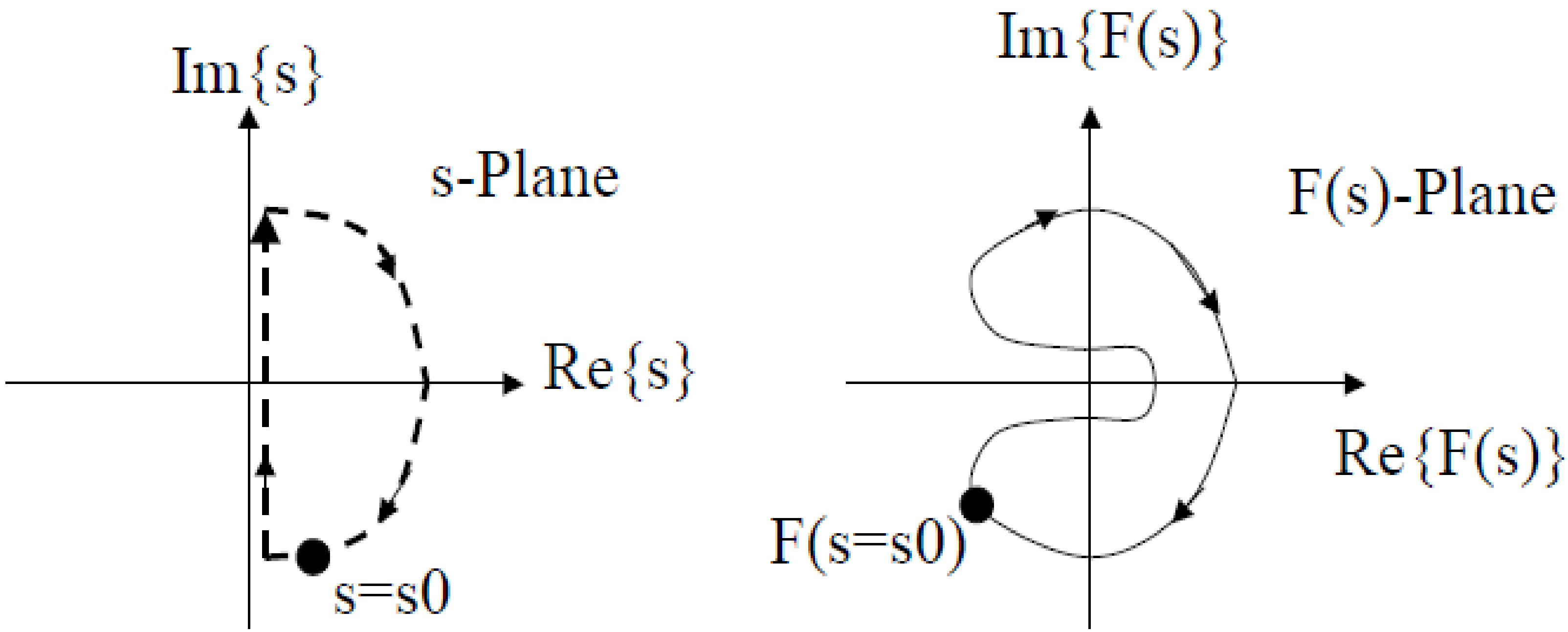




Imagine F(s=s0) to be a handle of a crank winding a spring…

F(s)-Plane Representation
Useful to Find Net Phase Change, N

- Define N = Number of CW encirclements of origin, in the F(s)-Plane Plot.



Also note N can be negative – corresponding to CCW encirclements.

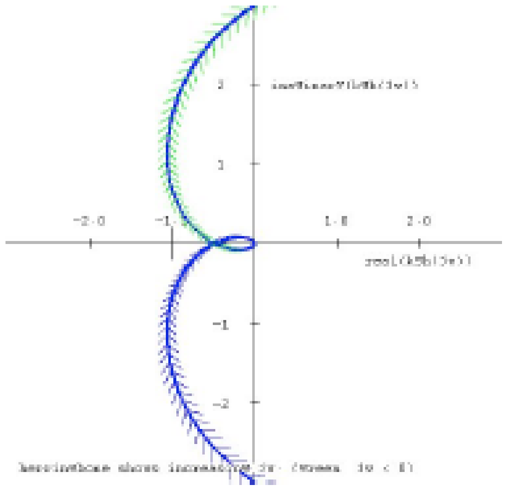
KGH(s)-Plane More Convenient To Determine Stability

- Note: KGH(s) = F(s) – 1.
- Hence plot of KGH(s) is shifted version of F(s)-Plane plo
- N is determined by number of CW encirclements of –1
- Nyquist Stability Theorem (Formally stated)
  - If P=0 then stable iff no encirclements of –1.
  - If P!=0 then stable iff Z = P + N = 0
- Procedure:
  1. Find the KGH(s)-Plot
  2. Examine plot, find N
  3. Examine factors of KGH(s), to find P = # RHP Poles of KGH(s)
  4. Z = P + N, Z = #RHP Poles of T(s)
  5. Stable iff Z = 0

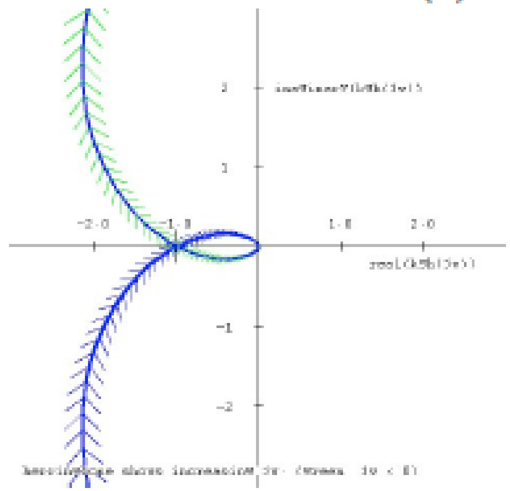
Note: Factors of KGH(s) typically easy to find, as open loop transfer function is usually built up from several cascaded (simpler) blocks.

Shape of Nyquist Plot Specific to Gain (K), Reveals Stability

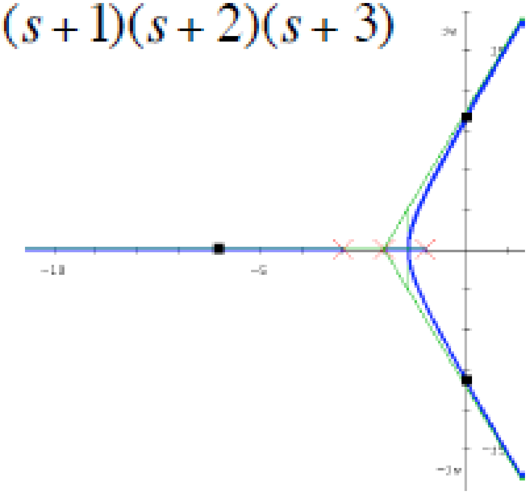
$$KGH(s) = \frac{K}{(s+1)(s+2)(s+3)}$$



K=30



K=60

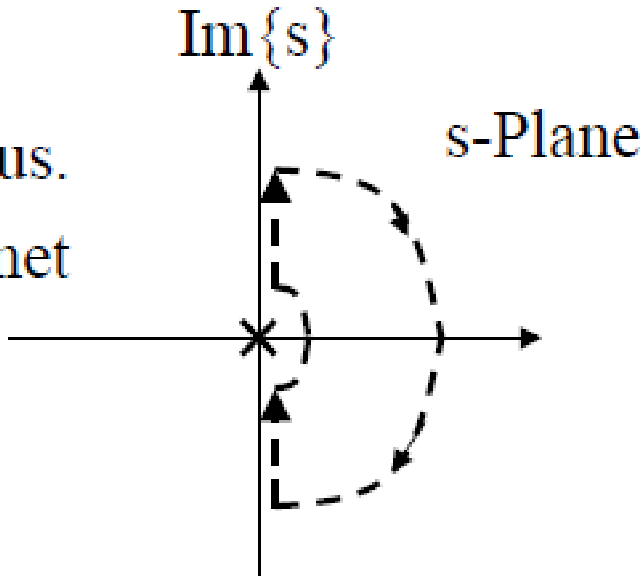


K=60

- With K>60 the Nyquist Plot encircles –1 point in the CW direction.
- (Alt Approaches: RL+Routh or Bode)

Exclude Poles/Zeros on jw Axis Except for Integrators

- Can't integrate over a pole – yields infinite (rude) result.
- Adjust contour in s-plane to move around poles and zeros. Use tiny radius.
- Exclusion eliminates contribution to net phase change of F(s).
- Typically not effecting # of encirclements of -1 point.
- Omitting cases with KGH(s) having poles on jw axis, other than origin…



Summary: Learning Objectives

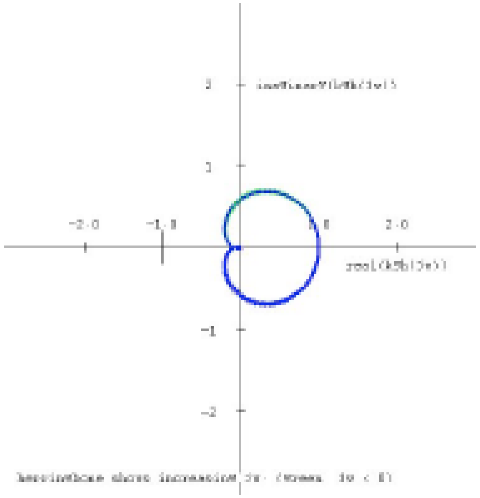
- Construct a Nyquist plot given KGH(s).
- Determine stability using a Nyquist plot:
  1. Find the KGH(s)-Plot
  2. Examine plot, find N = # CW encirclements of -1
  3. Examine factors of KGH(s), to find P = # RHP Poles
  4. Z = P + N
- Nyquist Stability Theorem (Formally stated)
  – If P=0 then stable iff no encirclements of –1.
  – If P!=0 then stable iff Z = P + N = 0
- Find Gain/Phase Margins given Nyquist plot.
  – GM: Increase in K necessary to scale plot to encircle –1.
  – PM: Rotation of plot CW needed to encircle –1.

Consider Limits When Plotting KGH(s)

$$KGH(s) = \frac{6}{(s+1)(s+2)(s+3)}$$

Limit	KGH(s)=?
jw -> +0	
jw -> -0	
jw -> +inf	
jw -> -inf	

- Note symmetry above & below real axis.
- Contribution to plot for |jw| -> infinity collapses to a single point.



Example

- Construction of Nyquist loci
  - Loop transfer function
  - By hand
    - Calculate features
      - Asymptotes: behavior as
      - Location of axes crossings

$$L(s) = \frac{25(s+1)}{s(s+2)\left(s^2+2s+16\right)}$$

$\omega \to 0$ and $\omega \to \infty$

- System loop

$$L(s)=\frac{25(s+1)}{s(s+2)\left(s^2+2s+16\right)}$$

– Construct Nyquist:

$$L(j\omega)=\frac{25(j\omega+1)}{j\omega(j\omega+2)\left(-\omega^2+2j\omega+16\right)}\,\square\,\frac{-j(-j\omega+2)\left(16-\omega^2-2j\omega\right)}{-j(-j\omega+2)\left(16-\omega^2-2j\omega\right)}$$

$$=\frac{-j25(j\omega+1)(-j\omega+2)\left(16-\omega^2-2j\omega\right)}{\omega\left(4+\omega^2\right)\left(\left(16-\omega^2\right)^2+4\omega^2\right)}$$

$$=\frac{-j25\left(2+\omega^2+j\omega\right)\left(16-\omega^2-2j\omega\right)}{\omega\left(4+\omega^2\right)\left(\omega^4-28\omega^2+256\right)}$$

$$L(j\omega) = \frac{25\omega\left(12-3\omega^{2}\right) - j\left(800+400\omega^{2}-25\omega^{4}\right)}{\omega\left(4+\omega^{2}\right)\left(\omega^{4}-28\omega^{2}+256\right)}$$

$$\operatorname{Re} L(j\omega) = \frac{25\left(12-3\omega^{2}\right)}{\left(4+\omega^{2}\right)\left(\omega^{4}-28\omega^{2}+256\right)}$$

$$\operatorname{Im} L(j\omega) = \frac{\left(25\omega^{4}-400\omega^{2}-800\right)}{\omega\left(4+\omega^{2}\right)\left(\omega^{4}-28\omega^{2}+256\right)}$$

- Quantitative analysis
  - Limits

$$L(s)=\frac{25(s+1)}{s(s+2)\left(s^2+2s+16\right)}$$

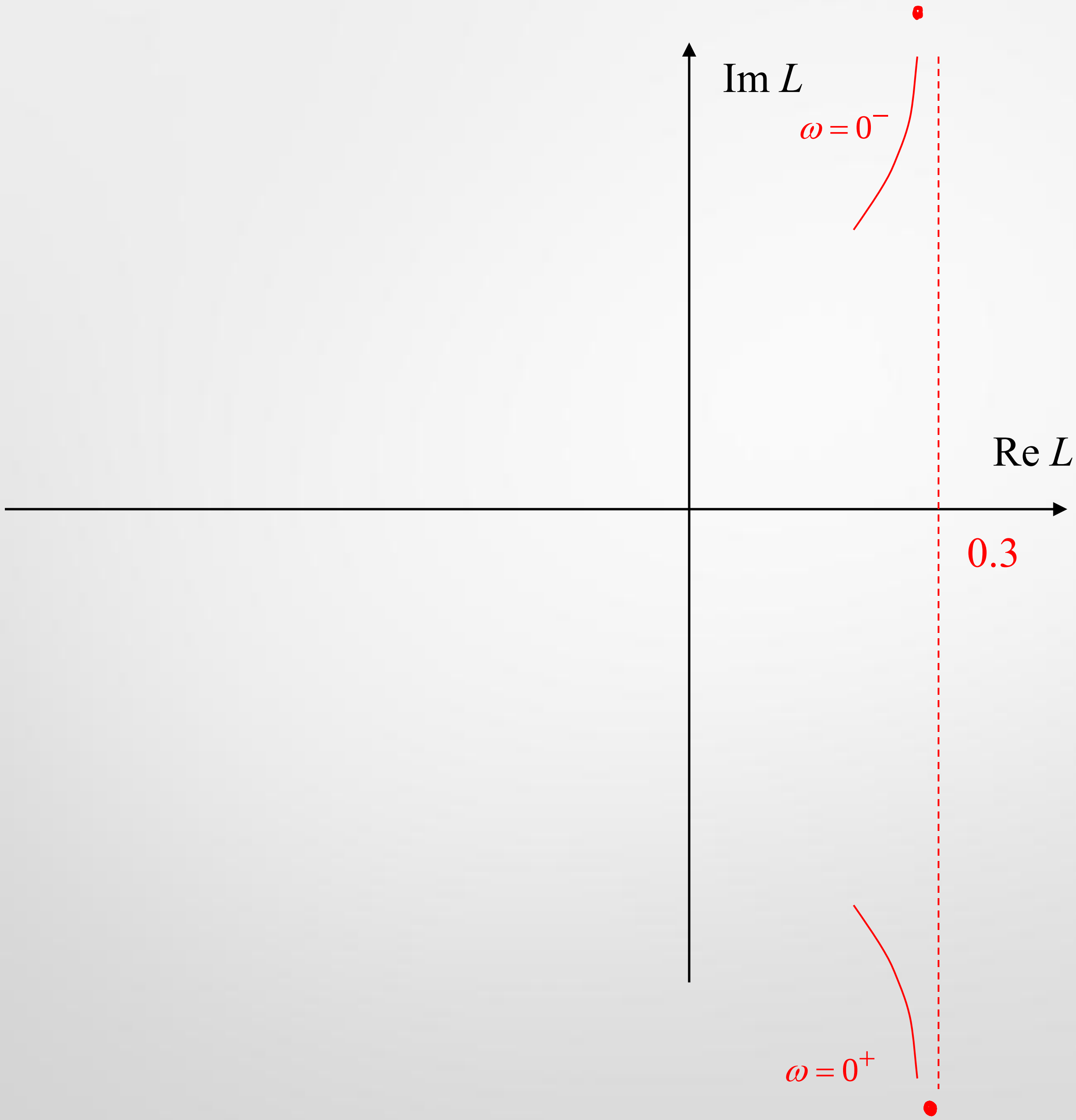
$$\operatorname{Re}L(j\omega)=\frac{25\left(12-3\omega^2\right)}{\left(4+\omega^2\right)\left(\omega^4-28\omega^2+256\right)}$$

$$\operatorname{Im}L(j\omega)=\frac{\left(25\omega^4-400\omega^2-800\right)}{\omega\left(4+\omega^2\right)\left(\omega^4-28\omega^2+256\right)}$$

$$\lim_{\omega\to 0}\operatorname{Re}L(j\omega)=\frac{300}{(4)(256)}=0.293$$

$$\lim_{\omega\uparrow 0}\operatorname{Im}L(j\omega)=\frac{-800}{\omega(4)(256)}=+\infty$$

$$\lim_{\omega\downarrow 0}\operatorname{Im}L(j\omega)=\frac{-800}{\omega(4)(256)}=-\infty$$


Im L
$\omega = 0^-$
Re L
0.3
$\omega = 0^+$

- Asymptotes for large ω
  - Keep dominant terms

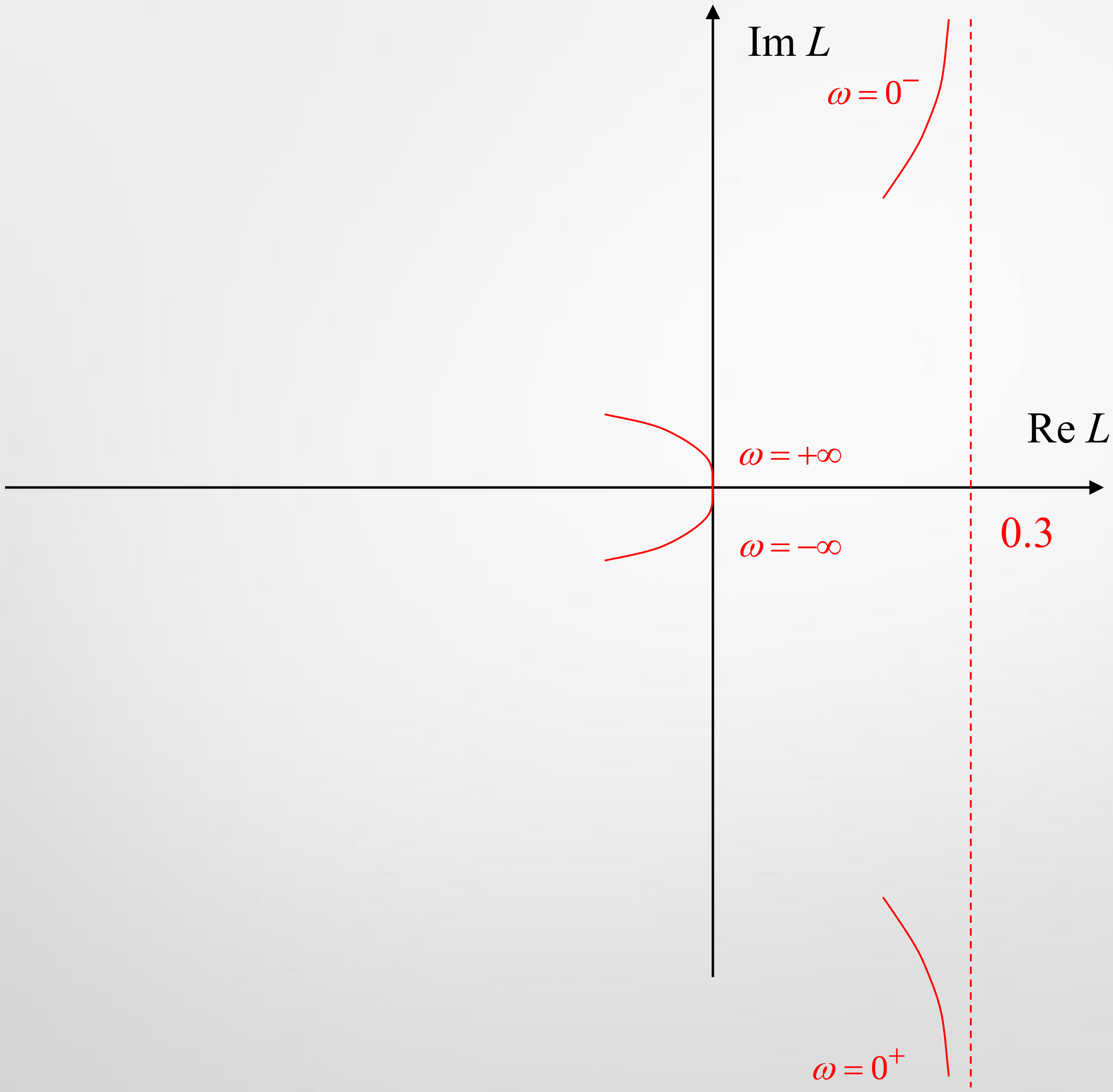
$$L(s)=\frac{25(s+1)}{s(s+2)(s^2+2s+16)}$$

$$\operatorname{Re} L(j\omega)=\frac{25(12-3\omega^2)}{(4+\omega^2)(\omega^4-28\omega^2+256)}$$

$$\operatorname{Im} L(j\omega)=\frac{(25\omega^4-400\omega^2-800)}{\omega(4+\omega^2)(\omega^4-28\omega^2+256)}$$

$$L(j\omega)\approx-\frac{75}{\omega^4}+j\frac{25}{\omega^3}$$

– For ω positive: + imaginary axis
– For ω negative: – imaginary axis



Im L
$\omega = 0^-$
Re L
$\omega = +\infty$
$\omega = -\infty$
0.3
$\omega = 0^+$

$$\operatorname{Re} L(j\omega) = \frac{25\left(12 - 3\omega^2\right)}{\left(4 + \omega^2\right)\left(\omega^4 - 28\omega^2 + 256\right)}$$

$$\operatorname{Im} L(j\omega) = \frac{\left(25\omega^4 - 400\omega^2 - 800\right)}{\omega\left(4 + \omega^2\right)\left(\omega^4 - 28\omega^2 + 256\right)}$$

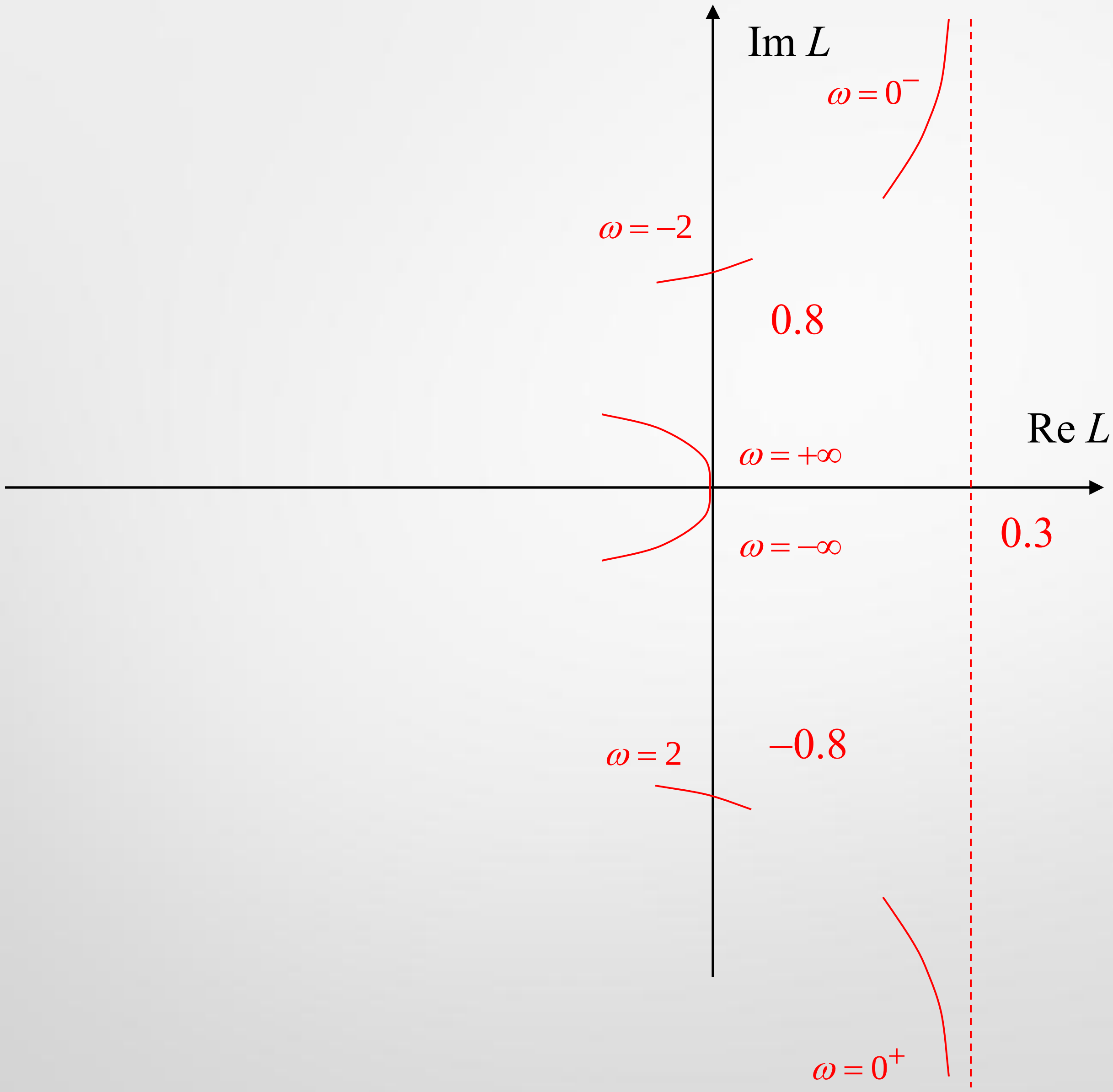
– Imaginary axis crossing(s)

- Real part = 0

$$\operatorname{Re} L(j\omega) = 0 = 300 - 75\omega^2$$

$$\omega = \pm 2$$

$$\operatorname{Im} L(j\omega)\Big|_{\omega=2} = \frac{25 \cdot 16 - 400 \cdot 4 - 800}{2 \cdot 8 \cdot \left(16 - 28 \cdot 4 + 256\right)} = -0.7815$$



Im L
$\omega = 0^-$
$\omega = -2$
0.8
Re L
$\omega = +\infty$
0.3
$\omega = -\infty$
$\omega = 2$
−0.8
$\omega = 0^+$

– Real axis crossing(s)

- Imaginary part = 0

$$\operatorname{Re} L(j\omega) = \frac{25\left(12 - 3\omega^2\right)}{\left(4 + \omega^2\right)\left(\omega^4 - 28\omega^2 + 256\right)}$$

$$\operatorname{Im} L(j\omega) = \frac{\left(25\omega^4 - 400\omega^2 - 800\right)}{\omega\left(4 + \omega^2\right)\left(\omega^4 - 28\omega^2 + 256\right)}$$

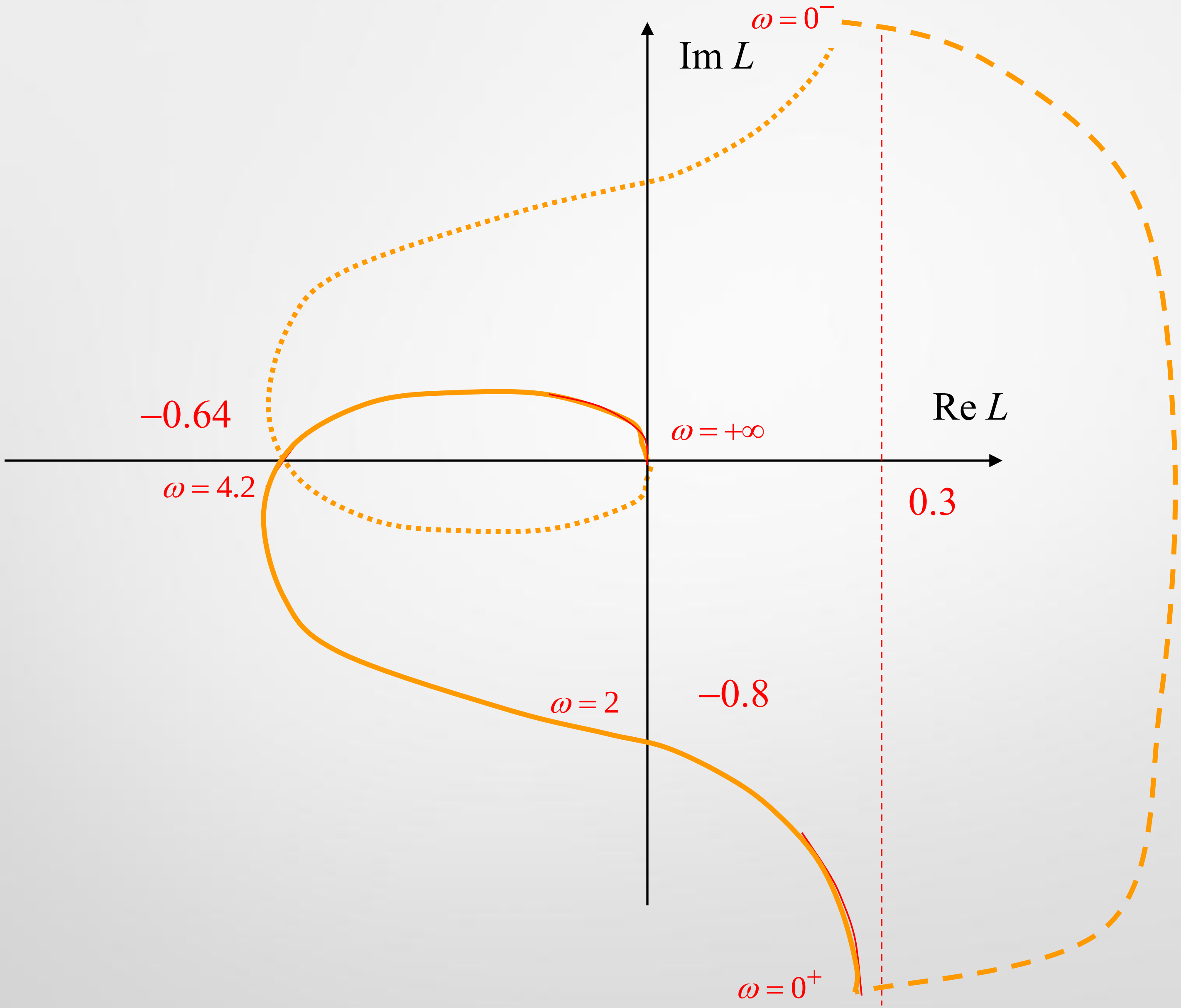
$$0 = \operatorname{Im} L(j\omega) = 25\omega^4 - 400\omega^2 - 800$$

$$= \omega^4 - 16\omega^2 - 32$$

$$\Rightarrow \omega^2 = 8 \pm \sqrt{64 + 32} = 4\left(2 \pm \sqrt{6}\right)$$

$$\Rightarrow \omega = \sqrt{4\left(2 + \sqrt{6}\right)} = \pm 4.22$$

$$\operatorname{Re} L(j4.22) = -0.638$$



ω = 0⁻
Im L
Re L
–0.64
ω = +∞
ω = 4.2
0.3
–0.8
ω = 2
ω = 0⁺

- Matlab: Nyquist